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Gain Scheduled H_∞ Controllers for a Two Link Flexible Manipulator

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The dynamics of a two link flexible manipulator are highly nonlinear and vary with the configuration of the second link over the operating range of the manipulator. This fact makes it difficult for a single linear time invariant controller to achieve the desired closed-loop specifications. This paper investigates the use of robust gain scheduled H_∞ controllers for endpoint trajectory tracking of a two link flexible manipulator with the angular position of the second link used as a scheduling parameter. A new method of gain scheduling with guaranteed stability at all frozen intermediate operating points is proposed. The method exploits the explicit and observer based compensator form of H_∞ controllers resulting from the normalized coprime factor robust stabilization approach. Simulation results corresponding to straight line motion of the tip of the second link are shown to illustrate the use of the proposed method.

I. Introduction

THE design of controllers for large flexible structures has been the subject of considerable research. Many authors have focused on control of single link flexible manipulators.[1] Some have extended this work to two link manipulators with a rigid upper arm and a flexible forearm.[2] Very little research has been done on modeling[3] and control[4,5] of two link manipulators with both links flexible.

For both rigid and flexible manipulators model-based control has proven to be effective in trajectory tracking. Open-loop control based on the solution of the inverse dynamics problem has been used to position single[1] and multilink flexible manipulators.[6] Such open-loop control strategies with torque inputs computed off line have been found to be attractive when a relatively accurate analytical model is available. To further improve the performance of these open-loop techniques, a sensor-based feedback is usually added. Paden et al.[7] have used feed-forward based on the solution of the inverse dynamics problem together with local proportional derivative (PD) controllers on joint angle errors. A collocated proproportional integral derivative (PID) controller and a noncollocated linear quadratic Gaussian (LQG)-based controller[4] have also been used.

The dynamics of a two link flexible manipulator are highly nonlinear and vary over the operating range of the manipulator. This fact makes it difficult for a single linear time invariant controller to achieve the desired closed-loop design specifications. This paper investigates the use of controller scheduling to avoid this problem. Each linear time invariant controller is designed to cover a certain range of the motion of the manipulator. The use of a linear time invariant controller over a range of motion of the manipulator requires robust stability of the closed-loop system against plant parameter variations. Robustness with respect to model uncertainties is achieved by using the H_∞ controller synthesis method. The standard H_∞ synthesis method based on the state space solution technique of Glover and Doyle[8] provides a powerful tool for the design of multivariable robust linear time invariant controllers. Performance goals and closed-loop frequency response specifications are incorporated into the problem as frequency dependent weighting functions.

An alternative approach uses the normalized coprime factor description of the plant. Unlike the standard approach, this method involves no iteration and gives an exact and explicit solution to the robust stabilization problem. Performance goals are incorporated by using loop shaping design procedure.[9]

This paper is organized as follows: First we present a linear time (parameter) varying analytical model of a two link flexible manipulator followed by a brief discussion of the continuity of the map from parameter space to the space of parameterized controllers. We describe the normalized coprime factor robust stabilization problem and exploit its explicit solution to propose a method for selecting operating points at which linear time invariant controllers are designed. The main result is presented in Sec. VI where a new procedure for scheduling observer-based state feedback controllers for guaranteed closed-loop stability at all frozen intermediate operating points is proposed. The method is applied to a two link flexible manipulator and simulation results are presented to illustrate the use of the proposed technique.

II. Nonlinear Model and Linearization

In this section the equations of motion of a two link manipulator with both links flexible is described. The links are assumed to be subjected to only bending deformations in the plane of motion, and torsional effects are neglected. The links are modeled as Euler-Bernoulli beams of uniform density with clamped-free boundary conditions. The equations of motion are derived using standard Lagragian formulation and the assumed modes method. In this method the bending variables are expanded as a series of mode shapes, and it is usually sufficient to consider the first few bending modes in the model. In this paper we consider one bending mode together with the rigid body mode for each link resulting in four generalized coordinates. Figure 1 depicts a two link flexible manipulator in a general deformed configuration and the variables used in the derivation of the equations of motion. The motion of the manipulator is assumed to take place in a horizontal plane, and hence gravity has no effect.

The dynamic equations of motion of a two link manipulator model can be written as

$$M(q)\ddot{q} + C(q, \dot{q}) = Bu \tag{1}$$

where $q \triangleq [\theta_1 \;\; \theta_2 \;\; \eta_1 \;\; \eta_2]^T$ is the vector of generalized coordinates (one joint angle and one modal amplitude for each link), $M(q)$ is a positive definite symmetric inertia matrix, $C(q, \dot{q})$ is a vector of

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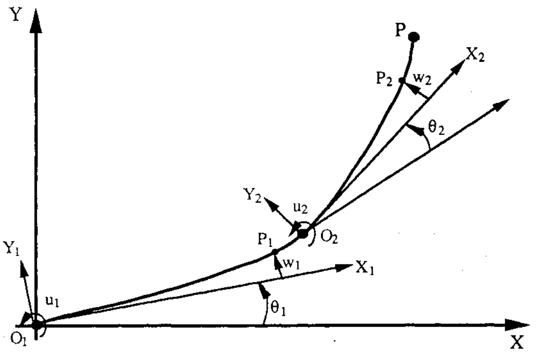

Fig. 1 Planar two link flexible manipulator.

centrifugal and coriolis forces, B is the input matrix, and $u \triangleq [u_1\ u_2]^T$ is a vector of input torques. The nonlinear equation of motion can be linearized about a nominal trajectory of the corresponding rigid link manipulator $[q_n(t),\ \dot{q}_n(t)]$ with $q_n \triangleq [\theta_{1n}\ \theta_{2n}\ 0\ 0]^T$. Let $\delta q \triangleq q - q_n$, $x(t) \triangleq [\delta q^T\ \delta\dot{q}^T]^T$, $M_n \triangleq M(q_n)$ and let $y \triangleq [X_p\ Y_p]^T$ be the position of a point P at the end of the second link. If $u = u_n + \delta u$ and $y = y_n + \delta y$, then Eq. (1) reduces to

$$\begin{aligned} \dot{x}(t) &= A_n(t)x(t) + B_n(t)\delta u(t) + W_n(t) \\ \delta y(t) &= C_n(t)x(t) \end{aligned} \tag{2}$$

where

$$A_n(t) \triangleq \begin{bmatrix} 0 & I \\ -M_n^{-1}\left.\dfrac{\partial C}{\partial q}\right|_{q_n} & -M_n^{-1}\left.\dfrac{\partial C}{\partial \dot{q}}\right|_{q_n} \end{bmatrix}, \qquad B_n(t) \triangleq \begin{bmatrix} 0 \\ M_n^{-1}B \end{bmatrix} \tag{3}$$

$$C_n(t) \triangleq \begin{bmatrix} \left.\dfrac{\partial y}{\partial q}\right|_{q_n} & \left.\dfrac{\partial y}{\partial \dot{q}}\right|_{q_n} \end{bmatrix}, \qquad W_n \triangleq \begin{bmatrix} 0 \\ M_n^{-1}(Bu_n - F_n) \end{bmatrix} \tag{4}$$

The vector of input torques is the sum of a nominal feed-forward input u_n and a corrective feedback torque δu. If the desired trajectory is such that the high frequency content is negligible,[10] a good first approximation of the nominal input u_n is the vector of torques required to move the tip of the corresponding rigid manipulator along the desired trajectory. u_n is easily computed from the inverse dynamics of a rigid link manipulator.

A more accurate vector of input torques is obtained by solving the inverse dynamics problem of the flexible manipulator, which results in noncausal input torques. These noncausal input torques can be found in the frequency domain using an iterative procedure proposed by Bayo et al.[6] The general inversion problem, which includes the flexible manipulator control problem, has been solved in the context of nonlinear geometric control for nonlinear nonminimum phase systems.[11]

Although open-loop strategies work well in the presence of good mathematical models, they are usually used with some kind of stabilizing feedback. This use involves the choice of a corrective input δu in Eq. (2) such that the closed-loop system achieves the required robust stability and performance in the presence of plant model and input and output uncertainties.

Equation (2) is a linear time varying state space description of the perturbation plant model where the time dependence of the system matrices A_n, B_n, and C_n is implicit through the generalized coordinates. $W_n(t)$ is a known input that depends on the choice of the nominal feed-forward torque u_n. Since the reference trajectory is known for all time, Eq. (2) may be viewed as parameter varying with the joint angle θ_2 as a parameter. The variation of this parameter is usually slow as compared with the fast dynamics of the flexible modes. It is well known that if a system is stable for each parameter, then the parameter varying system is stable for sufficiently slow parameter variations.[12,13] This fact suggests the use of θ_2 (a state variable that can be measured on line) as a scheduling parameter and the following procedure for controller design:

1) Several operating points that cover the range of the plant's dynamics are selected, and a linear time invariant approximation of the plant model is obtained at each of these operating points.

2) A linear time invariant controller is designed for each of these linear approximations of the plant.

3) The controller gains are interpolated between operating points to obtain the overall linear time varying controller.

If a controller designed based on an operating point is to be used for a certain range around the operating point, the closed-loop system corresponding to the point must have robust stability and robust performance to guarantee acceptable stability and performance of the overall gain scheduled system. Stability and performance robustness against modeling errors, disturbance inputs and measurement noise can be achieved using an H_∞ optimal controller.

III. Uncertainty Description and Continuity of Controller Design

The H_∞ approach to optimal control design has given promising results in the area of robust stabilization of plants with unstructured uncertainties. Unstructured plant uncertainties are uncertainties about which there is no information available except an upper bound on the magnitude as a function of frequency. They are usually modeled as additive or multiplicative perturbations to the nominal plant.[9] It has been shown[14] that it is not possible to incorporate uncertainty in the number of right half-plane poles using these types of uncertainty descriptions. Another type of uncertainty description that overcomes this limitation is described as additive perturbations to the normalized coprime factors of a plant. If the nominal plant transfer function is written in terms of its left coprime factors as

$$G_n = M_n^{-1}N_n \tag{5}$$

where M_n and N_n are stable transfer matrices normalized such that

$$M_n M_n^* + N_n N_n^* = I \tag{6}$$

then a family of perturbed plants (M_n, N_n, ε) is given by

$$G_\varepsilon = \{(M_n + \Delta M)^{-1}(N_n + \Delta N) : \ \|\Delta M,\ \Delta N\|_\infty < \varepsilon\} \tag{7}$$

This description is particularly important in the analysis of feedback stability as it forms the bases for defining the Gap[15] and the Graph metrics[16] and the associated Graph topology.

The continuity of the map from parameter space to the space of parameterized closed-loop transfer functions is necessary for a parameter to be used as a scheduling signal. This property is established using the graph topology as follows.

Let $\lambda_0 \in \Re$ and suppose that $P_\lambda(s)$ is the transfer matrix of the system

$$\begin{aligned} \dot{x}(t) &= A_\lambda x(t) + B_\lambda \delta u(t) \\ \delta y(t) &= C_\lambda x(t) \end{aligned} \tag{8}$$

If $(A_{\lambda_0}, B_{\lambda_0}, C_{\lambda_0})$ is stabilizable and detectable and $(A_\lambda, B_\lambda, C_\lambda)$ is continuous in λ at $\lambda = \lambda_0$ in the usual norm topology on Euclidean space, then the map $\lambda \to P_\lambda$ is continuous in the graph topology at λ_0.[16]

Suppose an H_∞ controller, K_λ, is designed for the previous system. A careful analysis of the state space procedure for H_∞ controller design shows that the matrices A_k, B_k, C_k, and D_k of the resulting controller are continuous in the plant parameter λ in the usual norm topology. Hence the map $\lambda \to K_\lambda(s) \triangleq C_k(sI - A_k)^{-1}B_k + D_k$ is continuous in the graph topology.

Let K_{λ_0} stabilize P_{λ_0} and the maps $\lambda \to P_\lambda$, $\lambda \to K_\lambda$ be continuous at λ_0 in the graph topology. Then there exists a neighborhood N of λ_0 such that the closed-loop transfer matrix $H(P_\lambda, K_\lambda)$ is stable for all $\lambda \in N$. Moreover $H(P_\lambda, K_\lambda)$ is continuous at λ_0 in the norm topology.[16]

Because of the continuity of the designed controller and the closed-loop transfer matrix in λ, the latter can be used as a scheduling parameter. This use guarantees stability of the closed-loop system at each frozen point provided the controller designs are performed at sufficiently close operating points. If in addition the variation of the scheduling parameter is slow enough, then the overall gain scheduled system remains stable.

The robust stabilization problem is to find the largest positive number ε_{max} such that $G_{\varepsilon_{max}}$ in Eq. (7) is robustly stabilizable. In the standard H_∞ controller design, robust stability and performance objectives are incorporated into the generalized plant to form the interconnection structure, and the state space approach of Glover and Doyle is used to design a stabilizing controller. This approach gives a state space formula for a specific optimal controller called the central controller and parameterizes all suboptimal stabilizing controllers as a linear fractional transformation. In general the central controller cannot be written as exact observer-based state feedback. This controller has a structure of an observer-based compensator with an additional worst disturbance term.[8]

An alternative approach is the normalized coprime factor robust stabilization problem. It has been shown[9] that this problem has an explicit solution, and the resulting central controller can be written as an exact observer-based compensator, which makes it convenient for gain scheduling.

IV. Normalized Coprime Factor Robust Stabilization

Let the nominal plant model be written in terms of its normalized left coprime factors as given by Eq. (5). If ΔM and ΔN are stable but unknown transfer functions, then

$$G = (M_n + \Delta M)^{-1}(N_n + \Delta N) \tag{9}$$

represents a perturbed plant. The normalized coprime factor robust stabilization approach addresses the problem of designing a controller K that stabilizes not only the nominal plant but also the family of perturbed plants defined by Eq. (7). The solution to the normalized coprime factor robust stabilization problem is given by the following two lemmas.[9]

Lemma 1: Let (M, N) be a normalized left coprime factorization of G. Then the feedback system (M, N, K, ε) is robustly stable if and only if (G, K) is stable and

$$\left\| \begin{bmatrix} K(I-GK)^{-1}M^{-1} \\ (I-GK)^{-1}M^{-1} \end{bmatrix} \right\|_\infty \le \varepsilon^{-1} \triangleq \gamma \tag{10}$$

Equivalently (M, N, ε) is robustly stabilizable if and only if

$$\inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} K(I-GK)^{-1}M^{-1} \\ (I-GK)^{-1}M^{-1} \end{bmatrix} \right\|_\infty \le \varepsilon^{-1} \tag{11}$$

where the infimum is taken over all stabilizing controllers K.

This problem can be written in the standard form for an H_∞ optimization problem, and the solution is given by the following lemma.

Lemma 2: Let (M, N) be a normalized left coprime factorization of G. Then

$$\inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} K(I-GK)^{-1}M^{-1} \\ (I-GK)^{-1}M^{-1} \end{bmatrix} \right\|_\infty = (1 - \|M, N\|_H^2)^{-0.5} \tag{12}$$

where $\|\cdot\|_H$ is the Hankel norm.

The previous lemma gives an explicit expression for the maximum stability margin

$$\varepsilon_{max} = (1 - -\|M, N\|_H^2)^{0.5} \tag{13}$$

and hence avoids the need for iteration. If in addition $G = C(sI - A)^{-1}B$ and A^* is the conjugate transpose of A, then the resulting central controller is given by

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}\delta y \\ \delta u &= F\hat{x} \end{aligned} \tag{14}$$

where

$$\begin{aligned} F &\triangleq -B^*X \\ W &\triangleq I + (XZ - \gamma^2 I) \\ \hat{A} &\triangleq A + BF + \gamma^2 W^{*-1}ZC^*C \\ \hat{B} &\triangleq -\gamma^2 W^{*-1}ZC^* \end{aligned} \tag{15}$$

and X and Z solve the following control and filter algebraic Riccati equations:

$$\begin{aligned} A^*X + XA - XBB^*X + C^*C &= 0 \\ AZ + ZA^* - ZC^*CZ + BB^* &= 0 \end{aligned} \tag{16}$$

If $L \triangleq \gamma^2 W^{*-1}ZC^*$, then Eq. (14) can be written as

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + B\delta u + L(C\hat{x} - \delta y) \\ \delta u &= F\hat{x} \end{aligned} \tag{17}$$

which has an exact observer-based compensator structure. Since A, B, and C are state matrices of the system (and hence known), it makes sense to interpolate the state feedback and observer gains only making this particular controller structure convenient for gain scheduling.

An H_∞ controller designed using the standard approach tends to cancel the slow poles of lightly damped systems. Hence a small uncertainty in a pole location may result in undesirable closed-loop behavior. The normalized coprime factor robust stabilization method avoids this problem[17] and leads to a simple and explicit formula for a stabilizing controller that maximizes robustness. Unlike the standard H_∞ problem, the normalized coprime factor robust stabilization problem does not address performance directly. To influence performance, the loop shaping design procedure proposed by McFarlane and Glover is used.[14] This procedure is based on the fact that the maximum singular values of the closed-loop transfer function can be manipulated (over an appropriate frequency range) by shaping the singular values of the corresponding open-loop transfer function.

In this method the plant is first shaped using pre- and/or post-compensators to alter the singular values of the nominal plant. A controller K_∞ that robustly stabilizes the shaped plant is designed. The final controller K is then obtained by combining the controller K_∞ with the compensators.

V. Loop Shaping and Selection of Design Points

In this section we outline a procedure for incorporating performance objectives into the normalized robust stabilization problem and the use of its explicit solution in the selection of operating points at which linear time invariant controllers are designed. The state space description of the linearized parameter varying model is given by

$$\begin{aligned} \dot{x} &= A_\lambda x + B_\lambda \delta u + W_\lambda \\ \delta y &= C_\lambda x \end{aligned} \tag{18}$$

where λ is the parameter. As mentioned earlier the controller design is continuous in the parameter. Moreover, if the controller is designed based on the normalized coprime factorization framework, it has perfect observer-based state feedback form, and the parameterized observer and state feedback gains can be interpolated to obtain a parameter varying controller. Performance objectives can also be incorporated using loop shaping.

Loop shaping involves the selection of an appropriate weight W_s to form the shaped plant G_s. If, for example, W_s is used as a precompensator, then $G_s = G_n W_s$. The shaped plant is then written in terms of its normalized coprime factors as

$$G_s = M_s^{-1} N_s \tag{19}$$

Using the explicit state space formulas given by Eqs. (14) and (15), a controller K_∞ is designed to stabilize the shaped plant with a specified stability margin $\varepsilon < \varepsilon_{\max} = \gamma_{\min}^{-1}$. The actual controller K is obtained by combining K_∞ with the shaping function as $K = W_s K_\infty$.

Each controller should be robust with respect to modeling errors resulting from the variation of the parameter. This known error can be taken care of by designing controllers at close enough operating points. If only modeling error due to parameter variation is considered, the $k+1$st design point may be selected such that

$$\|N_k - N_{k+1},\ M_k - M_{k+1}\|_\infty < \varepsilon_{k,\max} \tag{20}$$

where $G_k = M_k^{-1} N_k$, $G_{k+1} = M_{k+1}^{-1} N_{k+1}$ and $\varepsilon_{k,\max}$ is the maximum stability margin at the kth design point.

As explained in Sec. II only the first flexible mode of each link is included in the model. The modeling error introduced by neglecting higher frequency dynamics is usually included in the standard H_∞ controller synthesis as an additive uncertainty ΔG generally given by a high pass weighting function.

$$G = G_n + \Delta G \tag{21}$$

where ΔG is a function of frequency. This type of uncertainty can be included in the framework of coprime factors by considering it as a perturbation on either of the factors. For simplicity we consider the perturbed plant to be given by

$$G = M_n^{-1}(N_n + \Delta_N) \tag{22}$$

Comparing Eqs. (21) and (22), we have

$$\Delta G = M_n^{-1} \Delta_N \tag{23}$$

and the corresponding additive perturbation to N becomes

$$\Delta_N = M_n \Delta G \tag{24}$$

This type of uncertainty transformation can be very conservative, especially if the system has lightly damped poles, since ΔG becomes large at the resonant frequencies. In the case of flexible structures this conservatism can be reduced by taking advantage of the fact that damping increases with frequency. Assuming a constant damping ratio ζ for all modes, it can be shown that the peaks in the magnitude plot of the open-loop system decrease at about 40 dB/decade. This allows the use of additive perturbation ΔG, which rolls off at the same rate at high frequencies. Equation (24) can now be included in the selection of design points by modifying Eq. (20) to

$$\|N_k - N_{k+1},\ M_k - M_{k+1}\|_\infty < \varepsilon_{k,\max} - \|\Delta_N\|_\infty \tag{25}$$

VI. Scheduling

As mentioned earlier, widely varying dynamics of the two link flexible manipulator require some form of gain scheduling. Scheduling by interpolation of the gain matrices[18] or the poles and zeros of the controller[19] comes with no guarantee of the stability and performance of the system at intermediate points. More work needs to be done in this direction.

As explained in Sec. IV, the normalized coprime factor robust stabilization problem results in an exact observer-based state feedback compensator

$$\begin{aligned} \dot{\hat{x}} &= A_\lambda \hat{x} + B_\lambda \delta u + L_\lambda (C_\lambda \hat{x} - \delta y) \\ \delta u &= F_\lambda \hat{x} \end{aligned} \tag{26}$$

In this paper we exploit this special form of the controller and the freedom we have in choosing a realization for a given transfer function to propose a procedure for scheduling the observer and controller feedback gains with guaranteed closed-loop stability at all frozen intermediate operating points. The construction of this procedure requires the following theorems, which rely on the fact that a given transfer matrix can have many minimal realizations each related to another by a similarity transformation. The proofs of the theorems are given in the Appendix.

Theorem 1: Suppose A_0 and A_1 are two stability matrices and $Q = Q^T > 0$ is given. If P_0 and P_1 are the symmetric positive definite solutions of

$$\begin{aligned} A_0^T P_0 + P_0 A_0 &= -Q \\ A_1^T P_1 + P_1 A_1 &= -Q \end{aligned} \tag{27}$$

respectively, and

$$T_1 \triangleq (P_1)^{-1/2} (P_0)^{1/2} \tag{28}$$

then the interpolated matrix

$$A_\lambda = (1-\lambda) A_0 + \lambda T_1^{-1} A_1 T_1, \qquad \lambda \in [0, 1] \tag{29}$$

is a stability matrix.

The previous theorem defines a similarity transformation T_1 and an alternative realization $T_1^{-1} A_1 T_1$, which results in a stable interpolated transfer matrix. Note that multiplying Q by any positive scalar results in the same transformation T_1 and therefore has no effect on A_λ. The following important theorem defines a scheduling law for the feedback gain matrix so that the closed-loop system remains stable at all frozen intermediate points.

Theorem 2: Let (A_0, B_0, C_0) and (A_1, B_1, C_1) be minimal realizations of a parameterized transfer function P_λ at two values of the parameter. Let F_0 and F_1 be state feedback gains such that $A_{F0} \triangleq A_0 + B_0 F_0$ and $A_{F1} \triangleq A_1 + B_1 F_1$ are stability matrices. Let P_{F0} and P_{F1} be the symmetric positive definite solutions of

$$\begin{aligned} A_{F0}^T P_{F0} + P_{F0} A_{F0} &= -Q \\ A_{F1}^T P_{F1} + P_{F1} A_{F1} &= -Q \end{aligned} \tag{30}$$

for a given $Q = Q^T > 0$. Define.

$$\begin{aligned} T_{F1} &\triangleq (P_{F1})^{-1/2} (P_{F0})^{1/2} \\ \bar{F} &\triangleq (1-\lambda) F_0 + \lambda F_1 T_{F1} \\ \bar{A}_F &\triangleq (1-\lambda) A_{F0} + \lambda T_{F1}^{-1} A_{F1} T_{F1}, \qquad \lambda \in [0, 1] \end{aligned} \tag{31}$$

Let T_F be the solution of the Lyapunov equation

$$A_\lambda T_F - T_F \bar{A}_F = -B_\lambda \bar{F} \tag{32}$$

where $(A_\lambda, B_\lambda, C_\lambda)$ is a minimal realization of plant transfer function at some intermediate operating point. If $F_\lambda = \bar{F}T_F^{-1}$, then $A_\lambda + B_\lambda F_\lambda$ is a stability matrix for all $\lambda \in [0, 1]$.

Remark: Eq. (32) has a unique solution if and only if A_λ and $-\bar{A}_F$ have no common eigenvalues. In particular if A_λ has no eigenvalues in the right half of the complex plane (which is the case in flexible structures), Eq. (32) has a unique solution. Theorem 2 assumes the invertibility of this unique solution. It can be shown that if (A_λ, B_λ) is uncontrollable, then T_F is singular. In the extreme uncontrollable case when $B_\lambda = 0$ or when $\bar{F} = 0$, the unique solution $T_F = 0$ and therefore F_λ is not defined. In this paper we assume T_F to be invertible.

Corollary: Let (A_0, B_0, C_0) and (A_1, B_1, C_1) be minimal realizations of a parameterized transfer function P_λ at two values of the parameter. Let L_0 and L_1 be observer gains such that $A_{L0} \triangleq A_0 + L_0C_0$ and $A_{L1} \triangleq A_1 + L_1C_1$ are stability matrices. Let P_{L0} and P_{L1} be the symmetric positive definite solutions of

$$\begin{aligned} A_{L0}^T P_{L0} + P_{L0} A_{L0} &= -Q \\ A_{L1}^T P_{L1} + P_{L1} A_{L1} &= -Q \end{aligned} \tag{33}$$

for a given $Q = Q^T > 0$. Define

$$T_{L1} \triangleq (P_{L1})^{-1/2} (P_{L0})^{1/2}$$

$$\bar{L} \triangleq (1 - \lambda)L_0 + \lambda T_{L1}^{-1} L_1 \tag{34}$$

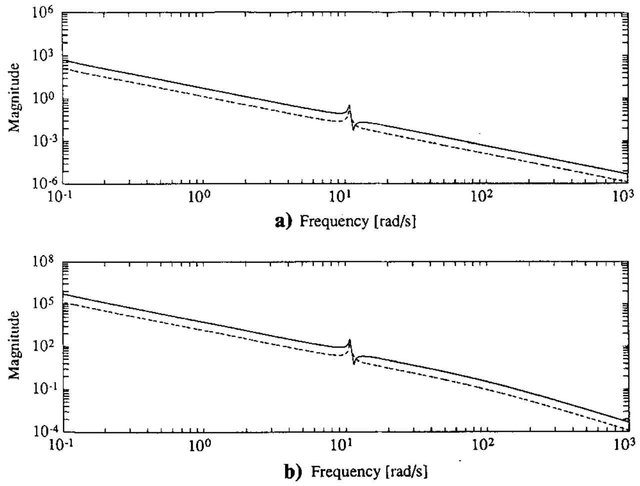

Fig. 2 Open-loop magnitude plot from the input u_1 to the tip coordinates (X_p, Y_p) for the a) unshaped and b) shaped plant.

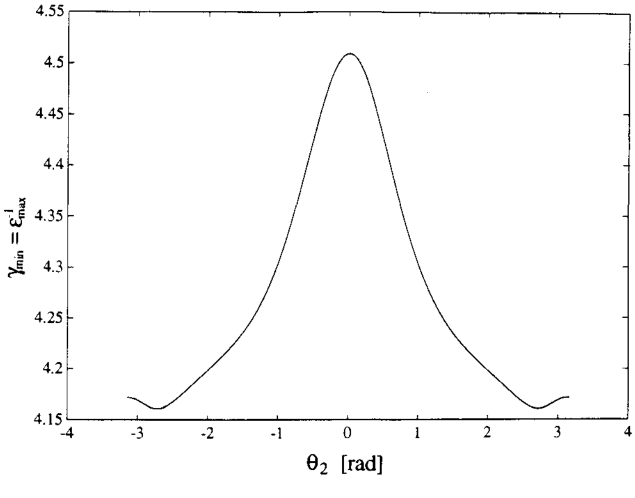

Fig. 3 Minimum achievable γ, $\gamma_{min} = \varepsilon_{max}^{-1}$.

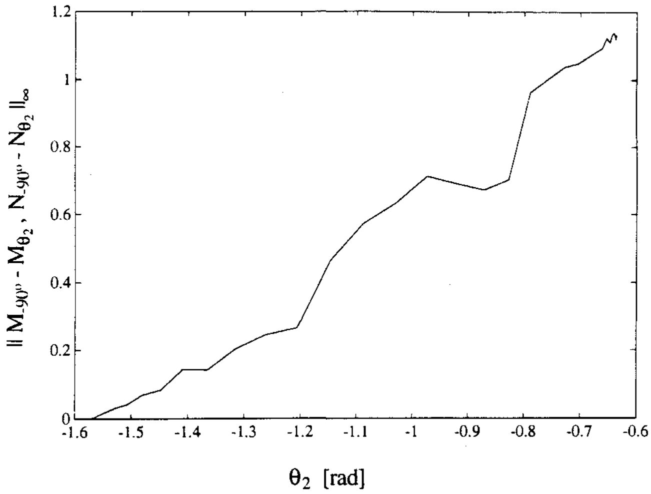

Fig. 4 Variation of the distance between plant transfer matrix at θ_2 and the transfer matrix at the first design point (θ_2 = 90 deg).

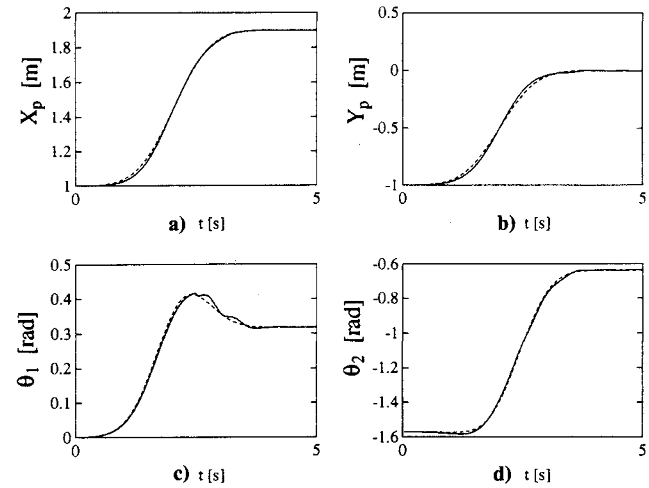

Fig. 5 Desired (solid) and achieved (dashed) tip position (X_p, Y_p) and joint angles (θ_1, θ_2) for the lower order model.

$$\bar{A}_L \triangleq (1 - \lambda)A_{L0} + \lambda T_{L1}^{-1} A_{L1} T_{L1}, \qquad \lambda \in [0, 1]$$

Let T_L be obtained from the solution of the Lyapunov equation

$$\bar{A}_L T_L^{-1} - T_L^{-1} A_\lambda = \bar{L} C_\lambda \tag{35}$$

If $L_\lambda = T_L \bar{L}$, then $A_\lambda + L_\lambda C_\lambda$ is a stability matrix for all $\lambda \in [0, 1]$.

Suppose $(A_\lambda, B_\lambda, C_\lambda)$ is a minimal realization of the open-loop transfer function P_λ parameterized by λ, and observer-based compensators are designed at the two extreme values of λ. The closed-loop system can be written as

$$\dot{z} = A_{c\lambda} z \tag{36}$$

where

$$A_{c\lambda} = \begin{bmatrix} A_\lambda + B_\lambda F_\lambda & -B_\lambda F_\lambda \\ 0 & A_\lambda + L_\lambda C_\lambda \end{bmatrix} \tag{37}$$

Theorem 2 and the previous corollary can now be used to define scheduling laws for F_λ and L_λ to guarantee the stability of $A_{c\lambda}$ for all $\lambda \in [0, 1]$. The resulting controller is parameterized by λ and is given by

$$\dot{\hat{x}}(t) = A_\lambda \hat{x} + B_\lambda \delta u + L_\lambda (C_\lambda \hat{x} - \delta y)$$

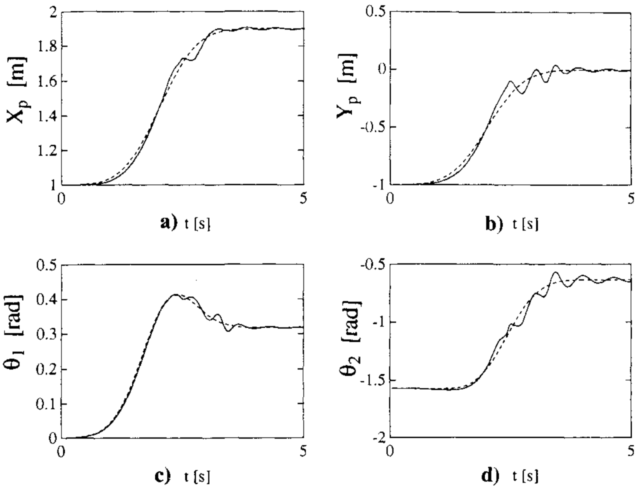

Fig. 6 Desired (solid) and achieved (dashed) tip position (X_p, Y_p) and joint angles (θ_1, θ_2) for the higher order evaluation model.

$$\delta u = F_\lambda \hat{x} \tag{38}$$

where

$$F_\lambda = [(1-\lambda)F_0 + \lambda F_1 T_{F1}]T_F^{-1} \tag{39}$$

$$L_\lambda = T_L[(1-\lambda)L_0 + \lambda T_{L1}^{-1}L_1] \tag{40}$$

VII. Simulations

In this section we apply the previous scheduling laws to design gain scheduled H_∞ controllers for a two link flexible manipulator with state space model given by Eqs. (2–4) and the following properties:

Length of links: $l_1 = l_2 = 1$ m
Cross-sectional area: $A_1 = A_2 = 6 \times 10^{-5}$ m^2
Bending stiffness: $(EI)_1 = (EI)_2 = 1.4\ N$ m^2
Mass per unit length: $\rho_1 = \rho_2 = 0.15$ kg /m

The tip of the second link is required to move in a straight line trajectory from $(X, Y) = (1, -1)$ to $(X, Y) = (1.9, 0)$. To minimize excitation of high frequency modes, the velocity profile of the tip was chosen to be Gaussian.[10]

The open-loop bode plot from the input torque u_1 at the base of the first link to the X_p and Y_p coordinates of the tip of the second link is shown in Fig. 2a for the nominal system at a typical operating point. It can be seen that the crossover frequency is around 1 rad/s. To increase the response time and the closed-loop bandwidth, a low pass compensator can be used. Since the order of the controller increases with the order of the compensator, a low order precompensator

$$W_s = \frac{s + 100{,}000}{s + 100} \tag{41}$$

is used to shape the plant. The magnitude plot of the shaped plant is shown in Fig. 2b where the crossover frequency has been increased to about 30 rad/s.

Figure 3 shows the variation of $\gamma_{\min}(\varepsilon_{\max}^{-1})$ with the scheduling parameter θ_2. The "distance" $\|\Delta N, \Delta M\|_\infty$ of the plant transfer matrix at θ_2 from the transfer matrix at the first design point ($\theta_2 = -90$ deg) is shown in Fig. 4. This plot can be used in conjunction with Fig. 3 to select operating points at which linear controller designs are performed. Unmodeled high frequency dynamics can be taken care of by using Eq. (25) in design point selection. Three operating points ($\theta_2 = -90$, -65, and -36 deg) were selected, and Eqs.(14) and (15) were used to design an H_∞ suboptimal controller ($\gamma_{\min} < \gamma = 4.6$) at each of these points. It should be pointed out that the use of either of the three linear time invariant controllers with the time varying plant resulted in an unstable closed-loop system suggesting some form of gain scheduling.

Equations (39) and (40) were used to compute the gains at intermediate operating points for $\theta_2 \in (-90, -65$ deg). The final position ($\theta_2 = -65$ deg) is then used as an initial position to compute the gains in the interval $\theta_2 \in (-65, -36$ deg). The performance of the closed-loop system with the rigid body torque as feed-forward and the gain scheduled controllers in the feedback loop is shown in Fig. 5. The X_p and Y_p coordinates of the tip of the second link and the joint angles θ_1 and θ_2 are shown, respectively, in figures *a*, *b*, *c*, and *d*. To see the effect of unmodeled higher frequency modes, the same controller was used to control a 12-state nonlinear evaluation model, which includes the second flexible mode of each link in addition to the first mode. Figure 6 shows the performance of the gain scheduled controllers with the higher order plant model. It can be seen that the additional mode has no effect on the stability and a small effect on the performance of the system indicating the robustness of the designed controller.

VIII. Conclusion

A new method of scheduling observer-based compensators for slowly time varying plants has been proposed. Unlike previously reported scheduling techniques, the proposed method results in guaranteed closed-loop stability at all frozen intermediate operating points. The problem of guaranteed performance at intermediate points is not addressed here and is a subject of further research. The method is applied to design gain scheduled H_∞ controllers for a two link flexible manipulator. The angular position of the second link was used as a scheduling parameter. The observer-based state feedback form of the compensators resulting from a normalized coprime factor robust stabilization approach was exploited to schedule the observer and state feedback gains. Simulation results corresponding to a straight line motion of the tip of the second link were presented to illustrate the use of the proposed method.

Appendix

Proof of theorem 1: It suffices to show that $\exists\, P_\lambda = P_\lambda^T > 0$ such that

$$A_\lambda^T P_\lambda + P_\lambda A_\lambda < 0 \qquad \forall \lambda \in [0, 1]$$

Choose $P_\lambda = P_0$. Then

$$A_\lambda^T P_\lambda + P_\lambda A_\lambda = [(1-\lambda)A_0^T + \lambda T_1^T A_1^T T_1^{-T}]P_0$$

$$+ P_0[(1-\lambda)A_0 + \lambda T_1^{-1} A_1 T_1]$$

$$= (1-\lambda)(A_0^T P_0 + P_0 A_0) + \lambda(T_1^T A_1^T T_1^{-T} P_0 + P_0 T_1^{-1} A_1 T_1)$$

$$= -(1-\lambda)Q + \lambda T_1^T (A_1^T T_1^{-T} P_0 T_1^{-1} + T_1^{-T} P_0 T_1^{-1} A_1) T_1$$

But $T_1^{-T} P_0 T_1^{-1} = P_1$. Therefore

$$A_\lambda^T P_\lambda + P_\lambda A_\lambda = -(1-\lambda)Q + \lambda T_1^T (A_1^T P_1 + P_1 A_1) T_1$$

$$= -(1-\lambda)Q - \lambda T_1^T Q T_1$$

$$\triangleq -(1-\lambda)Q - \lambda Q_1$$

Since $Q_1 = Q_1^T > 0$, we have

$$A_\lambda^T P_\lambda + P_\lambda A_\lambda < 0$$

and the proof is complete.

Proof of theorem 2:

$$A_\lambda + B_\lambda F_\lambda = A_\lambda + B_\lambda \bar{F} T_F^{-1}$$

$$= A_\lambda + [T_F \bar{A}_F - A_\lambda T_F] T_F^{-1}$$

$$= T_F \bar{A}_F T_F^{-1}$$

It only remains to show that $\bar{A}_F$ is a stability matrix. This fact is easily established by applying theorem 1 with A_0 and A_1 replaced by A_{F0} and A_{F1}, respectively.

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